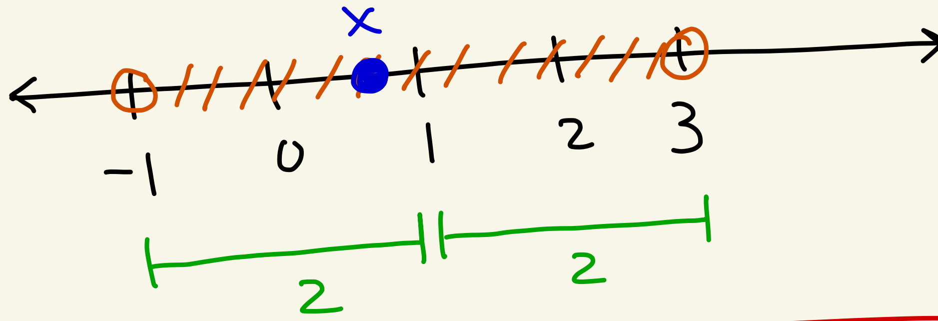
Math 4650
Homework 1
Solutions

1

(a) $|x-1| < 2$

$-2 < x-1 < 2$

$-1 < x < 3$

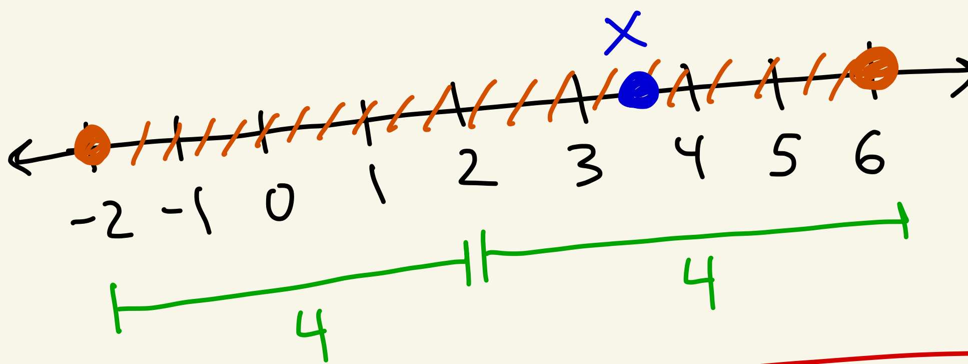


(b) $|-2+x| \leq 4$

$|x-2| \leq 4$

$-4 \leq x-2 \leq 4$

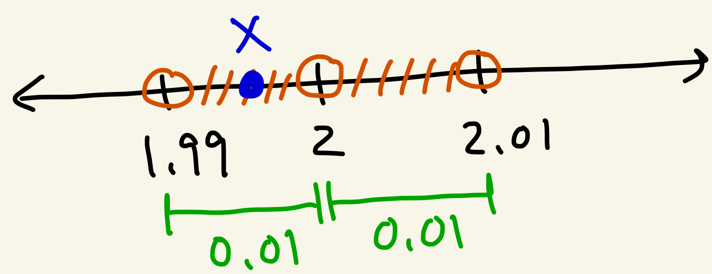
$-2 \leq x \leq 6$



(c) $0 < |x-2| < 0.01$

$-0.01 < x-2 < 0.01$ and $x \neq 2$

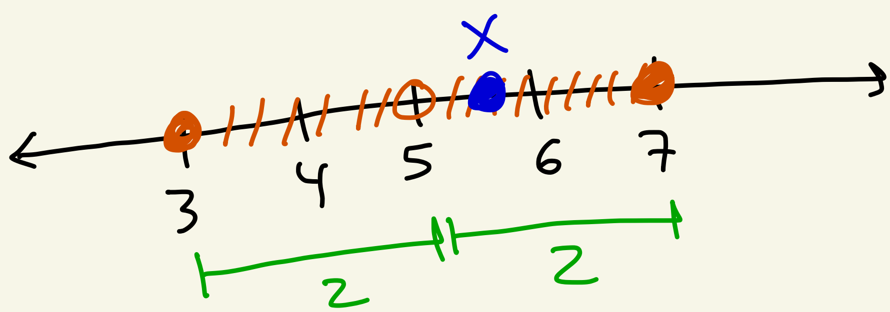
$1.99 < x < 2.01$ and $x \neq 2$



(d) $0 < |x-5| \leq 2$

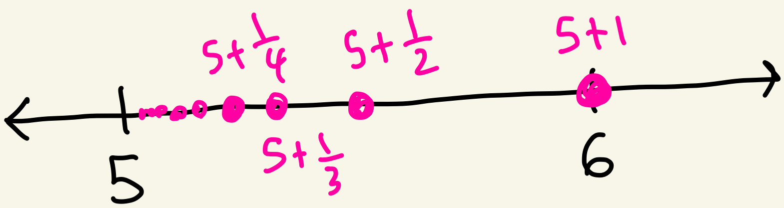
$-2 \leq x-5 \leq 2$ and $x \neq 5$

$3 \leq x \leq 7$ and $x \neq 5$



2

(a) $X = \left\{ 5 + \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\} = \left\{ 5+1, 5+\frac{1}{2}, 5+\frac{1}{3}, 5+\frac{1}{4}, \ldots \right\}$

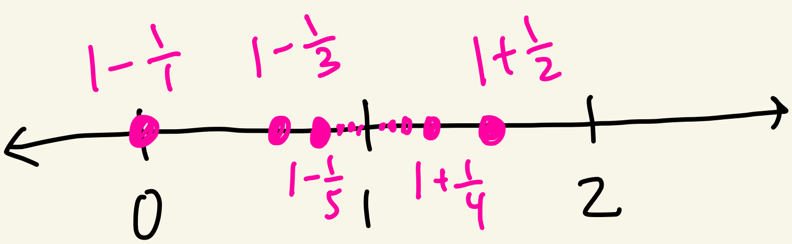


$\inf(X) = 5$

$\sup(X) = 6$

(b) $X = \left\{ 1 + \frac{(-1)^n}{n} \;\middle|\; n \in \mathbb{N} \right\}$

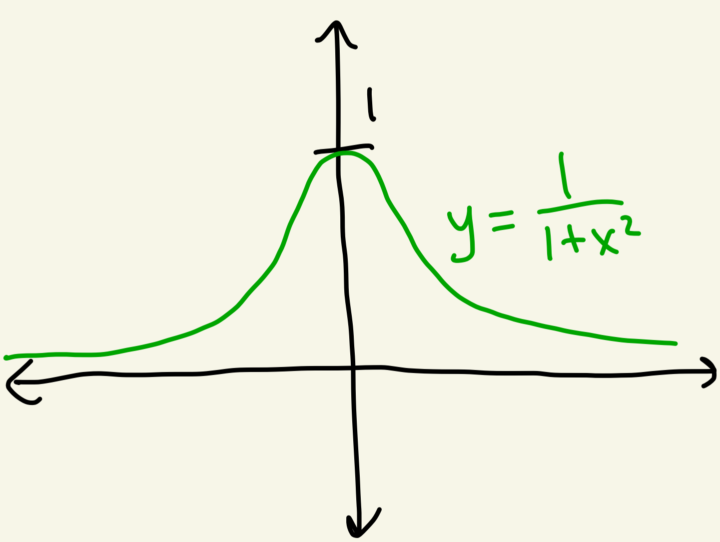
$= \left\{ 1-\frac{1}{1}, 1+\frac{1}{2}, 1-\frac{1}{3}, 1+\frac{1}{4}, 1-\frac{1}{5}, \ldots \right\}$



$\inf(X) = 0$

$\sup(X) = 1 + \frac{1}{2} = 1.5$

(c) $X = \left\{ \frac{1}{1+x^2} \;\middle|\; x \in \mathbb{R} \right\}$

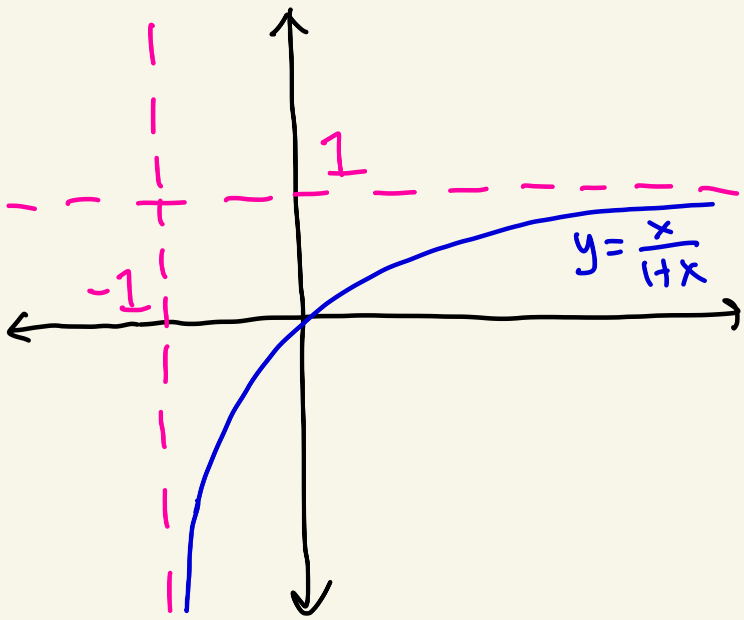


The set X consists of the y-values of this graph.

$\inf(X) = 0$

$\sup(X) = 1$

$\boxed{(d)}$ $X = \left\{ \frac{x}{1+x} \,\middle|\, x \in \mathbb{R} \text{ with } -1 < x \right\}$



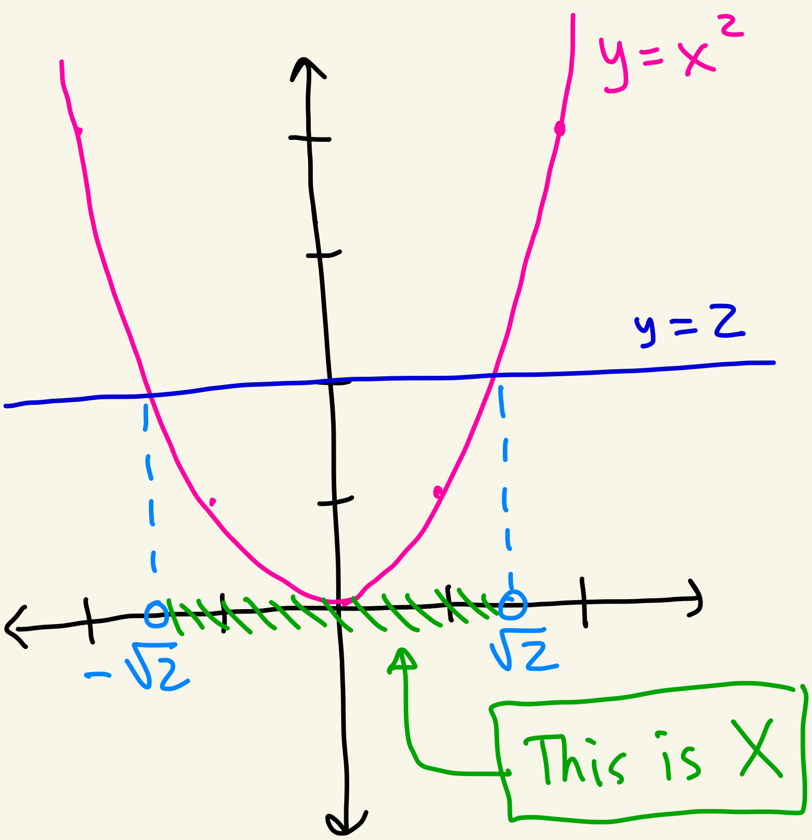
The set X consists of the y-values of this graph with $-1<x$

$\inf(X)$ does not exist

$\sup(X) = 1$

$\boxed{(e)}$ $X = \{ x \in \mathbb{R} \mid x^2 + 1 < 3 \}$

$= \{ x \in \mathbb{R} \mid x^2 < 2 \}$



We see from the picture that

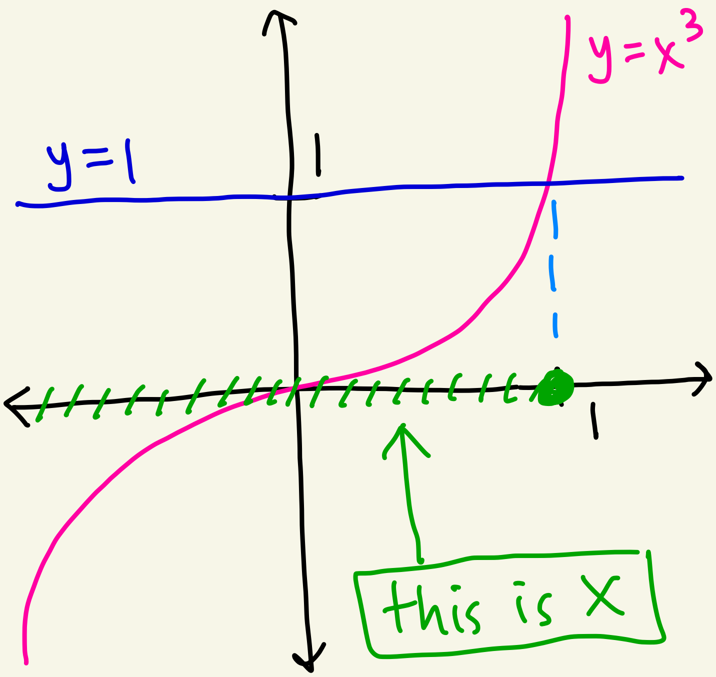
$X = \{ x \in \mathbb{R} \mid -\sqrt{2} < x < \sqrt{2} \}$

$= (-\sqrt{2}, \sqrt{2})$

$\inf(X) = -\sqrt{2}$

$\sup(X) = \sqrt{2}$

$\boxed{(f)}$ $X = \{ x \in \mathbb{R} \mid x^3 \leq 1 \}$



From the picture we see that

$X = \{ x \in \mathbb{R} \mid x \leq 1 \}$

$= (-\infty, 1]$

$\inf(X)$ does not exist

$\sup(X) = 1$

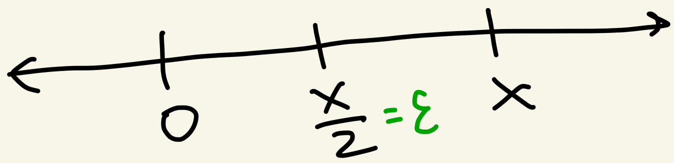
③ Let $x \in \mathbb{R}$ with $x \geq 0$.

Our assumption is that $x \leq \varepsilon$ for all $\varepsilon > 0$.

Let's show this implies that $x = 0$.

Suppose $x > 0$.

Then, $0 < \frac{x}{2} < x$.



Set $\varepsilon = \frac{x}{2}$

By assumption $x \leq \varepsilon$.

But then both $\varepsilon < x$ and $x \leq \varepsilon$ which is a contradiction.

Hence $x > 0$ cannot be true.

So, $x = 0$. $\blacksquare$

$\boxed{4}$

Suppose that a and b are both supremums for S.

Then a and b are both upper bounds for S.

Since a is a supremum for S and b is an upper bound for S, by def of supremum, we have that $a \leq b$.

a is a <u>least</u> upper bound for S

Since b is a supremum for S and a is an upper bound for S, by def of supremum we have that $b \leq a$.

b is a <u>least</u> upper bound for S

Since $a \leq b$ and $b \leq a$ we have that $a = b$. $\blacksquare$

(5)

We are given that b is an upper bound for S and that $b \in S$.

Let's show that $b = \sup(S)$.

(i) We already have that b is an upper bound for S.

(ii) Let's show that b is the least upper bound for S.

Let c be another upper bound for S.

Then, $x \leq c$ for all $x \in S$.

Since $b \in S$ this gives $b \leq c$.

Thus, b is the least upper bound for S.

By (i) and (ii), $b = \sup(S)$. $\blacksquare$

$\boxed{6 \text{ (a)}}$

Suppose A and B are non-empty subsets of $\mathbb{R}$ bounded from above and below.

Further assume that $A \subseteq B$.

Let $s_A = \sup(A)$ and $s_B = \sup(B)$.

Since s_B is an upper bound for B we know that $b \leq s_B$ for all $b \in B$.

This implies, because $A \subseteq B$, that $a \leq s_B$ for all $a \in A$.

Thus, s_B is an upper bound for A.

Since s_A is the least upper bound for A we know that $s_A \leq s_B$.

Thus, $\boxed{\sup(A) \leq \sup(B)}$.

A similar argument shows $\boxed{\inf(B) \leq \inf(A)}$.

You try.

Also, if $a \in A$, by def. we have $\boxed{\inf(A) \leq a \leq \sup(A)}$.

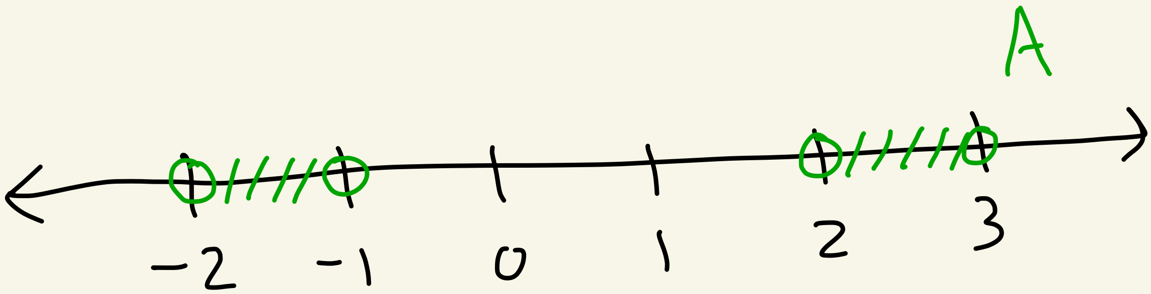
Hence, $\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)$

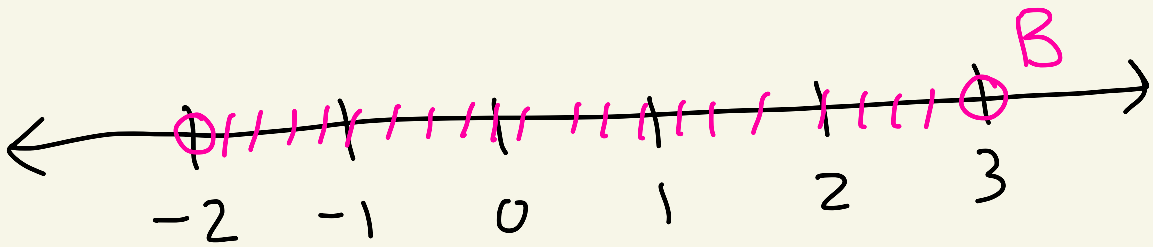
$\boxed{///}$

$\boxed{⑥ (b)}$

Set $A = (-2,-1) \cup (2,3)$

$B = (-2,3)$





Here $\inf(A) = -2 = \inf(B)$ and

$\sup(A) = 3 = \sup(B)$

but $A \neq B$

7(a) (We give two different proofs)

PROOF #1 – USING DEF OF SUPREMUM

proof: We are given that $s_A = \sup(A)$ and $s_B = \sup(B)$ exist. We are also given that $A \cap B \neq \phi$.

Since $A \cap B \subseteq A$ and $s_A = \sup(A)$ we know that $x \leq s_A$ for all $x \in A \cap B$

Since $A \cap B \subseteq B$ and $s_B = \sup(B)$ we know that $x \leq s_B$ for all $x \in A \cap B$.

Thus, s_A and s_B are upper bounds for $A \cap B$.

Thus, $A \cap B$ is bounded from above and $\sup(A \cap B)$ exists.

Let $m = \min\{s_A, s_B\}$.

Then from above m is an upper bound for $A \cap B$.

By the def of supremum (least upper bound) we know that $\sup(A \cap B) \leq m$.

Thus, $\sup(A \cap B) \leq \min\{\sup(A), \sup(B)\}$

$\blacksquare$

PROOF #2 – USING THE INF/SUP THEOREM

<u>proof</u>:

Let $s_A = \sup(A)$.

Since $A \cap B \subseteq A$ we know that $x \leq s_A$ for all $x \in A \cap B$.

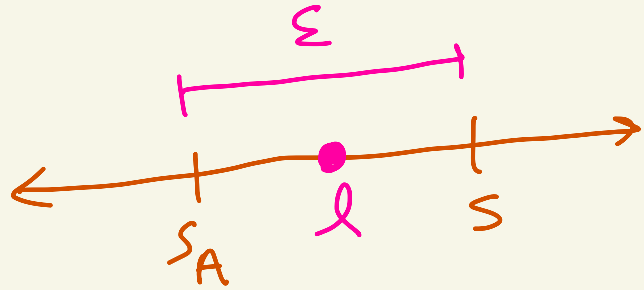
So, $A \cap B$ is bounded from above by s_A.

Thus, $s = \sup(A \cap B)$ exists.

Let's show that $s \leq s_A$.

Suppose that $s > s_A$.

Then, $\varepsilon = s - s_A > 0$.



By the inf/sup theorem since $s = \sup(A \cap B)$, there exists $\ell \in A \cap B$ with $\underbrace{s_A}_{s-\varepsilon} < \ell \leq s$.

But then $\ell \in A$ and $s_A < \ell$ which contradicts the fact that $s_A = \sup(A)$.

Therefore, $s \leq s_A$.

Similarly one can show that $s \leq s_B$.

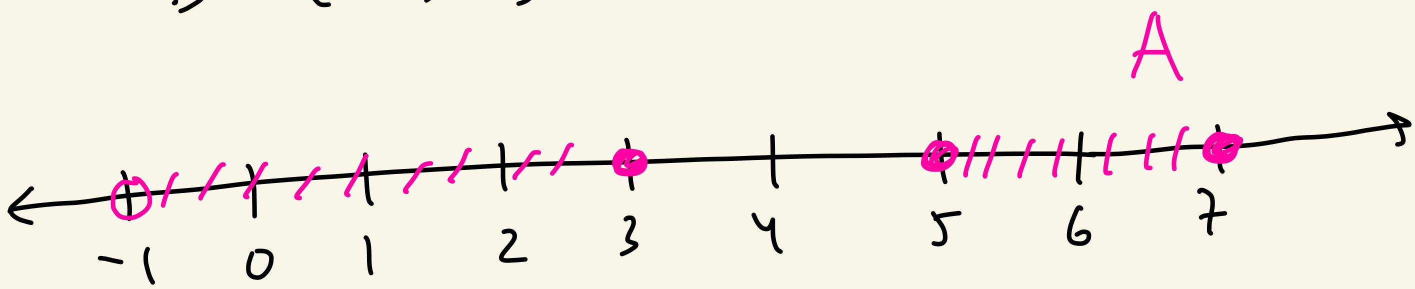
Thus, $s \leq \min\{s_A, s_B\}$

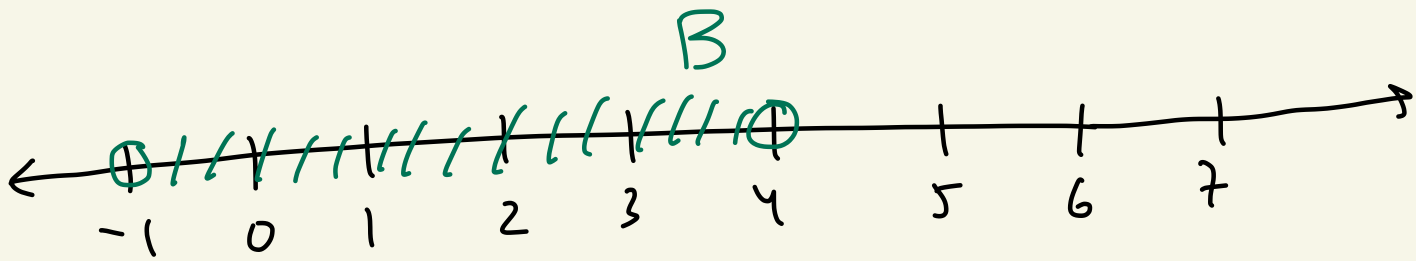
So, $\sup(A \cap B) \leq \min\{\sup(A), \sup(B)\}$. $\boxed{//}$

7 (b)

Let $A = (-1, 3] \cup [5, 7]$

$B = (-1, 4)$





Then, $A \cap B = (-1, 3]$

And $\sup(A) = 7$

$\sup(B) = 4$

$\min\{\sup(A), \sup(B)\} = 4$

$\sup(A \cap B) = 3$

Thus,

$\sup(A \cap B) \neq \min\{\sup(A), \sup(B)\}$

7(c) (We give two different proofs)

PROOF #1 – USING DEF OF SUPREMUM

proof:

Let $s_A = \sup(A)$ and $s_B = \sup(B)$.

We will assume that $s_A \leq s_B$.

If $s_B \leq s_A$, the same proof would work with A and B interchanged

Since $s_A \leq s_B$ we have that

$$s_B = \max\{\underbrace{\sup(A)}_{s_A}, \underbrace{\sup(B)}_{s_B}\}$$

We first show that s_B is an upper bound for $A \cup B$.

Let $x \in A \cup B$.

Then $x \in A$ or $x \in B$.

since $s_A = \sup(A)$

If $x \in A$, then $x \leq s_A \leq s_B$.

If $x \in B$, then $x \leq s_B$ since $s_B = \sup(B)$

Thus, no matter the case we have $x \leq s_B$.

So, s_B is an upper bound for $A \cup B$.

We will show that s_B is the least upper bound for $A \cup B$.

Suppose c is another upper bound for $A \cup B$.

Then $x \leq c$ for all $x \in A \cup B$.

Thus, $x \leq c$ for all $x \in B$.

So, c is an upper bound for B.

Since s_B is the least upper bound for B we get that $s_B \leq c$.

Thus, s_B is the least upper bound for $A \cup B$.

That is, $s_B = \sup(A \cup B)$.

So, $\sup(A \cup B) = \underbrace{\max\{\sup(A), \sup(B)\}}_{s_B}$

$\blacksquare$

PROOF #2 - USING THE INF/SUP THEOREM

proof:

Let $s_A = \sup(A)$ and $s_B = \sup(B)$.

We will assume that $s_A \leq s_B$.

If $s_B \leq s_A$, the same proof would work with A and B interchanged

Since $s_A \leq s_B$ we have that

$$s_B = \max\{\underset{\substack{\uparrow \\ s_A}}{\sup(A)}, \underset{\substack{\uparrow \\ s_B}}{\sup(B)}\}$$

We first show that s_B is an upper bound for $A \cup B$.

Let $x \in A \cup B$.

Then $x \in A$ or $x \in B$.

since $s_A = \sup(A)$

If $x \in A$, then $x \leq s_A \leq s_B$.

If $x \in B$, then $x \leq s_B$

since $s_B = \sup(B)$

Thus, no matter the case we have $x \leq s_B$.

So, s_B is an upper bound for $A \cup B$.

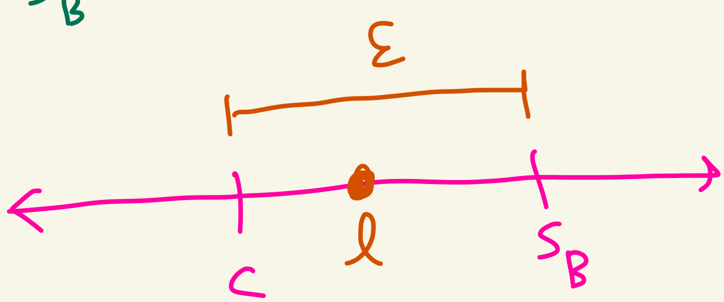
Now we show that s_B is the least upper bound for $A \cup B$.

Suppose that c is another upper bound for $A \cup B$.

We need to show that $s_B \leq c$.

Suppose that $s_B > c$.

Then $\varepsilon = s_B - c > 0$.



By the inf/sup theorem,
since $s_B = \sup(B)$, there exists $l \in B$
with $\underbrace{c}_{s_B - \varepsilon} < l \leq s_B$

Then $l \in A \cup B$ and $c < l$.

This contradicts the fact that c is

an upper bound for $A \cup B$.

Thus, $s_B > c$ can't be true.

So, $s_B \leq c$.

Thus, s_B is the least upper bound for $A \cup B$.

So, $s_B = \sup(A \cup B)$.

Thus, $\underbrace{\max\{\sup(A), \sup(B)\}}_{s_B} = \sup(A \cup B)$.

$\blacksquare$

8(a)

We break the proof into two cases.

Case 1: Suppose $a \le b$.

Then, $a-b \le 0$.

So, $|a-b| = -(a-b) = b-a$

Also, $b-a \geq 0$.

So, $|b-a| = b-a$

Thus, $|a-b| = |b-a|$.

Use: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Case 2: Suppose $a > b$

Then, $a-b > 0$.

So, $|a-b| = a-b$

Also, $b-a < 0$.

So, $|b-a| = -(b-a) = a-b$

Thus, $|a-b| = |b-a|$.

In both cases $|a-b| = |b-a|$. $\blacksquare$

$\boxed{8(b)}$ We break the proof into four cases and use: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Case 1: Suppose $a \geq 0$ and $b \geq 0$

Then, $ab \geq 0$.
So, $|ab| = ab$, $|a| = a$, and $|b| = b$.

Thus, $|ab| = ab = |a| \cdot |b|$

Case 2: Suppose $a \geq 0$ and $b < 0$

Then, $ab \leq 0$
So, $|ab| = -ab$, $|a| = a$, and $|b| = -b$.

Thus, $|ab| = -ab = a(-b) = |a| \cdot |b|$

Case 3: Suppose $a < 0$ and $b \geq 0$

Then, $ab \leq 0$
So, $|ab| = -ab$, $|a| = -a$, and $|b| = b$.

Thus, $|ab| = -ab = (-a)b = |a| \cdot |b|$

Case 4: Suppose $a < 0$ and $b < 0$

Then, $ab \geq 0$.
So, $|ab| = ab$, $|a| = -a$, and $|b| = -b$.

Thus, $|ab| = ab = (-a)(-b) = |a| \cdot |b|$

In all four cases we get $|ab| = |a| \cdot |b|$.

8 (c)

Use: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Using part (b) we get that

$$\left|\frac{a}{b}\right| = \left|a \cdot \frac{1}{b}\right| = |a| \cdot \left|\frac{1}{b}\right| \quad (*)$$

If $b \geq 0$, then $\frac{1}{b} \geq 0$ and $\left|\frac{1}{b}\right| = \frac{1}{b} = \frac{1}{|b|}$

If $b < 0$, then $\frac{1}{b} < 0$ and $\left|\frac{1}{b}\right| = -\left(\frac{1}{b}\right) = \frac{1}{-b} = \frac{1}{|b|}$

Thus, in either case $\left|\frac{1}{b}\right| = \frac{1}{|b|}$

Hence, $\left|\frac{a}{b}\right| = |a| \cdot \left|\frac{1}{b}\right| = |a| \cdot \frac{1}{|b|} = \frac{|a|}{|b|}$.

(*) from above

8 (d)

We have that

$$a < x < b \quad \text{and} \quad a < y < b$$

Thus

$$a < x < b \quad \text{and} \quad -b < -y < -a$$

Adding gives

$$a - b < x - y < b - a$$

Thus

$$-(b-a) < x - y < b - a$$

Hence

$$|x - y| < b - a$$

$\underline{c > 0}$:

$-c < z < c$ iff $|z| < c$

Note: $b - a > 0$ here since $b > a$

$\blacksquare$

8 (e)

Note that

$$|a| = |(a-b)+b| \leq |a-b| + |b|$$

triangle inequality

So,

$$\boxed{|a| - |b| \leq |a-b|} \quad (*)$$

Also,

$$|b| = |(b-a)+a| \leq |b-a| + |a|$$

So,

$$-|b-a| \leq |a| - |b|$$

From part (a) of this problem, $|a-b| = |b-a|$.

Thus, $\boxed{-|a-b| = -|b-a| \leq |a| - |b|} \quad (**)$

Hence, from $(*)$ and $(**)$ we get that

$$\boxed{-|a-b| \leq |a| - |b| \leq |a-b|}$$

Thus,

$$\big| |a| - |b| \big| \leq |a-b|$$

Here we use this fact from class: If $c \geq 0$, then $|x| \leq c$ iff $-c \leq x \leq c$

$\blacksquare$